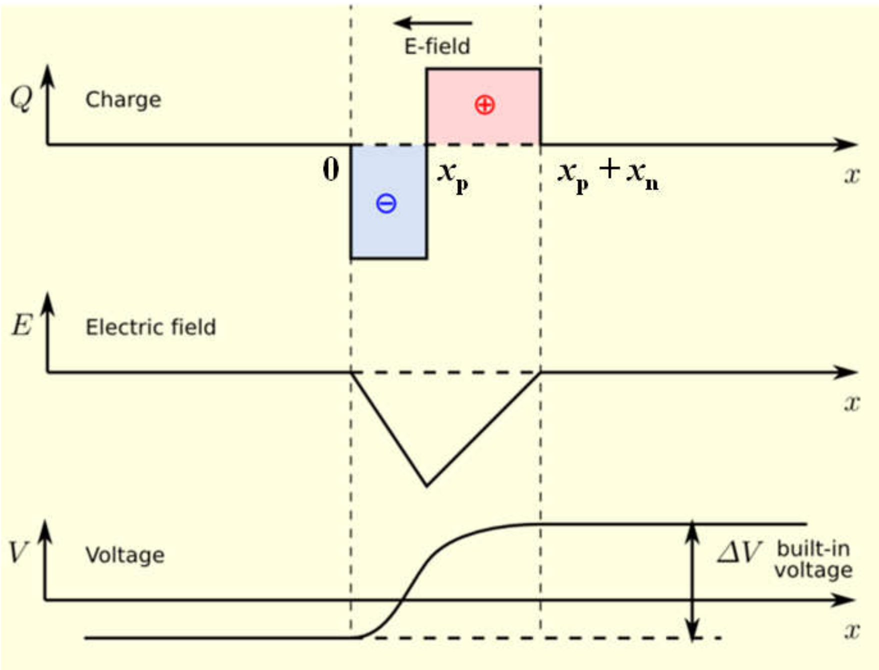
Depletion Width of a PN Junction

Consider a pn junction below. Doping concentrations are N_A and N_D in the p- and n-region, respectively. We apply the full depletion approximation and assume the abrupt transition at the pn junction. Assume the depletion widths for the p- and n-regions are x_p and x_n, respectively. The dielectric constant for this semiconductor is ε_s. The build-in potential is V_{bi}. Prove that the depletion width is

$$W = x_p + x_n = \sqrt{\frac{2\varepsilon_s}{q}\left(\frac{1}{N_A} + \frac{1}{N_D}\right)V_{bi}}$$



In the depletion region, the charge concentration $Q(x)$ is

$$\begin{cases} Q(x) = -N_A & \text{for } 0 < x < x_p \\ Q(x) = +N_D & \text{for } x_p < x < x_p + x_n \end{cases}$$

Outside the junction ($x < 0$ and $x > W$), $Q(x) = 0$

Based on the charge balance, we have

$$N_A x_p = N_D x_n$$

And because

$$W = x_p + x_n$$

We have

$$\begin{cases} x_p = \dfrac{N_D}{N_A + N_D} W \\ x_n = \dfrac{N_A}{N_A + N_D} W \end{cases}$$

Based on the Gauss's Law, we have

$$\frac{\partial E}{\partial x} = \frac{q}{\varepsilon_s} Q(x)$$

We can get the electric field $E(x)$

$$\begin{cases} E(x) = -\int_0^x \dfrac{q}{\varepsilon_s} N_A dx & \\ \quad = -\dfrac{q}{\varepsilon_s} N_A x & \text{for } 0 < x < x_p \\ E(x) = \int_0^x \dfrac{q}{\varepsilon_s} Q(x) dx & \\ \quad = -\int_0^{x_p} \dfrac{q}{\varepsilon_s} N_A dx + \int_{x_p}^x \dfrac{q}{\varepsilon_s} N_D dx & \\ \quad = -\dfrac{q}{\varepsilon_s} N_A x_p + \dfrac{q}{\varepsilon_s} N_D (x - x_p) & \text{for } x_p < x < x_p + x_n \end{cases}$$

And the electric field $E(x)$ is the gradient of the electric potential $V(x)$

$$\frac{\partial V}{\partial x} = -E(x)$$

We can get the potential $V(x)$

$$\begin{cases} V(x) = -\int_0^x \left(-\frac{q}{\varepsilon_s} N_A x \right) dx \\ \quad = \frac{1}{2} \frac{q}{\varepsilon_s} N_A x^2 & \text{for } 0 < x < x_p \\ V(x) = -\int_0^x E(x) dx \\ \quad = -\int_0^{x_p} \left(-\frac{q}{\varepsilon_s} N_A x \right) dx - \int_{x_p}^x \left(-\frac{q}{\varepsilon_s} N_A x_p + \frac{q}{\varepsilon_s} N_D (x - x_p) \right) dx \\ \quad = -\frac{1}{2} \frac{q}{\varepsilon_s} (N_A + N_D) x_p^2 + \frac{q}{\varepsilon_s} (N_A + N_D) x_p x - \frac{1}{2} \frac{q}{\varepsilon_s} N_D x^2 & \text{for } x_p < x < x_p + x_n \end{cases}$$

Here we assume $V(x = 0) = 0$, so $V(x = W = x_p + x_n) = V_{bi}$

We combine

$$V_{bi} = -\frac{1}{2} \frac{q}{\varepsilon_s} (N_A + N_D) x_p^2 + \frac{q}{\varepsilon_s} (N_A + N_D) x_p W - \frac{1}{2} \frac{q}{\varepsilon_s} N_D W^2$$

And

$$\begin{cases} x_p = \frac{N_D}{N_A + N_D} W \\ x_n = \frac{N_A}{N_A + N_D} W \end{cases}$$

We can solve

$$W = \sqrt{\frac{2\varepsilon_s}{q} \left(\frac{1}{N_A} + \frac{1}{N_D} \right) V_{bi}}$$